NOTES ON PERSISTENT HOMOLOGY

VAIBHAV KARVE

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1. Definitions

(1) Let $\mathbb{X}$: connected topological space, $f : \mathbb{X} \to \mathbb{R}$ be a continuous function. The sublevel sets of f form a 1-parameter family of nested subspaces $\mathbb{X}_a \subset \mathbb{X}_b$ whenever $a \leq b$.

(2) The *merge tree* denoted $G(f)$ is generated by increasing the threshold for f-values and keeping track of components. Components will merge but won't split. When merging, we preserve the older branch.

(3) Letting $a \leq b$ be two thresholds, denote by $\beta(a, b)$ the number of components in $\mathbb{X}_b$ that have a nonempty intersection with $\mathbb{X}_a$.
In the merge tree this is the number of subtrees with topmost points at value b that reach down to level a or below.

(4) **Filtrations.** K : simplicial complex, $f : K \to \mathbb{R}$ monotonic (which means f is non-decreasing along increasing chains of faces that is, $f(\sigma) \leq f(\tau)$ whenever σ is a face of τ). Monotonicity implies that the sublevel set $K(a) = f^{-1}(-\infty, a]$ is a subcomplex of K for every $a \in \mathbb{R}$.
Letting $a_1 < \cdots < a_n$ be the function values of the simplices in K and setting $a_0 = -\infty$, we get an increasing sequence of subcomplexes —

$$\emptyset = K_0 \subseteq \cdots \subseteq K_n = K$$

where $K_i = K(a_i)$, $\forall i$. This is called the *filtration* of f.

(5) Examples of filtrations : Čech complexes, alpha complexes, lower star filtration of a piecewise linear function.

(6) For every $i \leq j$ we have inclusion map $K_i \hookrightarrow K_j$ and therefore an induced homomorphism $f_p^{i,j} : H_p(K_i) \to H_p(K_j)$, for each dimension p.
The filtration thus results in a sequence of homology groups connected by homomorphisms —

$$0 = H_p(K_0) \to \cdots \to H_p(K_n) = H_p(K)$$

for each p : dimension.

(7) The p^{th} *persistent homology groups* are the images of the homomorphisms induced by the inclusion map. They are denoted $H_p^{i,j} = \text{im } f_p^{i,j}$ for $0 \leq i \leq j \leq n$. The corresponding p^{th} persistent Betti numbers are the ranks of these groups, $\beta_p^{i,j} = \text{rank} H_p^{i,j}$.

(8) Each persistent homology group $H_k^{i,j}$ consists of the homology classes of $K_i$that are still alive at K_j. Hence for every $i \leq j$ we have —

$$H_p^{i,j} = Z_p(K_i)/(B_p(K_j) \cap Z_p(K_i))$$

(9) Note that $H_p^{i,i} = H_p(K_i)$.

(10) Let γ be a class in $H_p(K_i)$. We say γ is *born at* K_i if $\gamma \notin H_p^{i-1,i}$. Ig γ is born at K_i then it *dies entering* K_j if it merges with an older class as we go from K_{j-1} to K_j, that is, $f_p^{i,j-1}(\gamma) \notin H_p^{i-1.j-1}$ but $f_p^{i,j}(\gamma) \in H_p^{i-1,j}$. Said differently, $\gamma \in H_p(K_i)$ is born at K_i if it is not in the image of $H_p(K_{i-1})$, that is, it is not in $H_p^{i-1,i}$. γ dies aentering K_j if that is the first time its image merges into the image of $H_p(K_{i-1})$.

(11) If γ is born at K_i and dies entering K_j then the *persistence* is defined to be

$$pers(\gamma) = a_j - a_i$$

that is, the difference in the function values. If γ is born at K_i but never dies then it has persistence ∞.