Interactive Theorem Proving in Lean

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Introduction

Lean: an Interactive Theorem Prover

An **interactive theorem prover** is a software that assists with the development of formal proofs.

- We used the **Lean theorem prover**, a toolkit that balances manual and automated theorem proving.
- Lean allows the user to define mathematical objects and proofs of statements about these objects, while Lean's language kernel checks them for accuracy.
- Traditionally, mathematicians use **set theory** (ZFC axioms) as the logical foundation for all their results. Lean however, uses **Type Theory** which is a richer and more expressive variant of set theory.
- The mathematical theory that makes theorem provers possible is the **Curry-Howard Isomorphism**. It equates programs with proofs and propositions with types.

Goal:

Formalize 3 different axiomatic geometry systems into the Lean theorem prover – **Euclid's**, **Hilbert's** and **Tarski's**.

Axiomatic Geometry

- 300 BCE : Euclid was the first to introduce the method of proving a geometrical result by applying an axiomatic system.
- 1899 : David Hilbert published *The Foundations of Geometry* to put geometry on a more rigorously foundation.
- 1959 : Alfred Tarski introduced the first set of geometry axioms that avoided Set Theory.

Figure 1: *Euclid of Alexandria, David Hilbert, and Alfred Tarski. source: Wikimedia Commons*

Primary Components of an Axiomatic System:

1. Primitives (undefined terms) are the most fundamental ideas with no intrinsic properties. These are defined as constants in Lean.
2. Axioms (postulates) are elementary statements about primitives that are assumed true, without need for proof. Leans allows us to declare axioms.
3. Propositions (theorems) are more complex statements that can be deduced from the axioms using mathematical logic. Lean calls requires a valid proof for these.

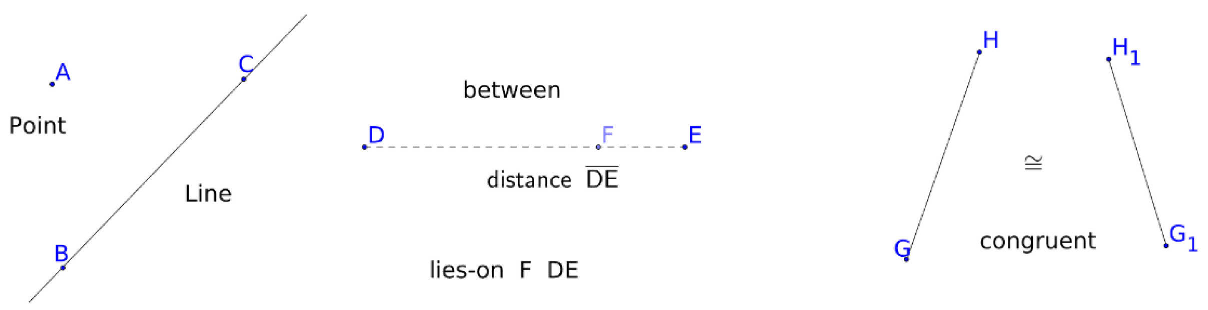

Figure 2: *Primitive notions (constants and structures in Lean) common to all three systems.*

Euclidean Geometry

Missing axioms

Euclid relied heavily on the behavior of geometry when drawn on a piece of paper to prove his propositions. All his constructions depended on a straightedge and compass. For example, he assumed that two circles intersect when one's center is the other's radius, but provided no justification for this fact. In order to formalize Euclid's postulates in Lean, we had to introduce several axioms that Euclid missed.

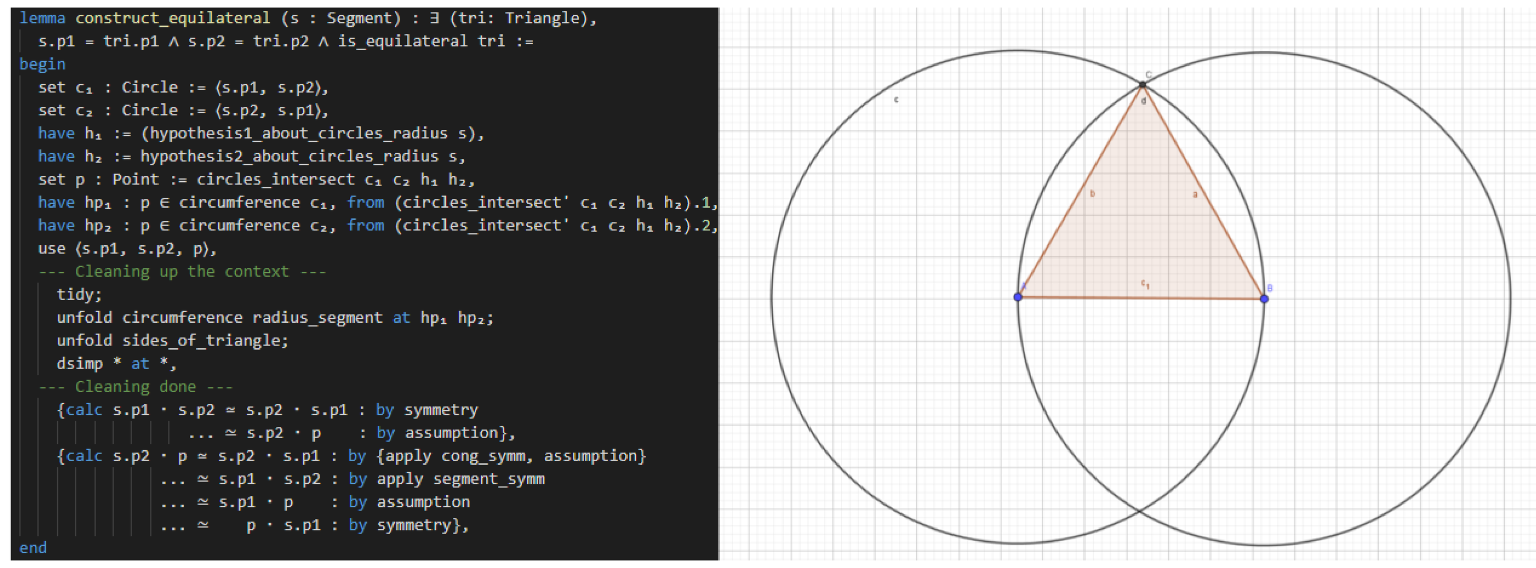

Figure 3: *(How to construct an equilateral triangle). An example of a simple proof expressed in Lean code as well as in a geometric picture.*

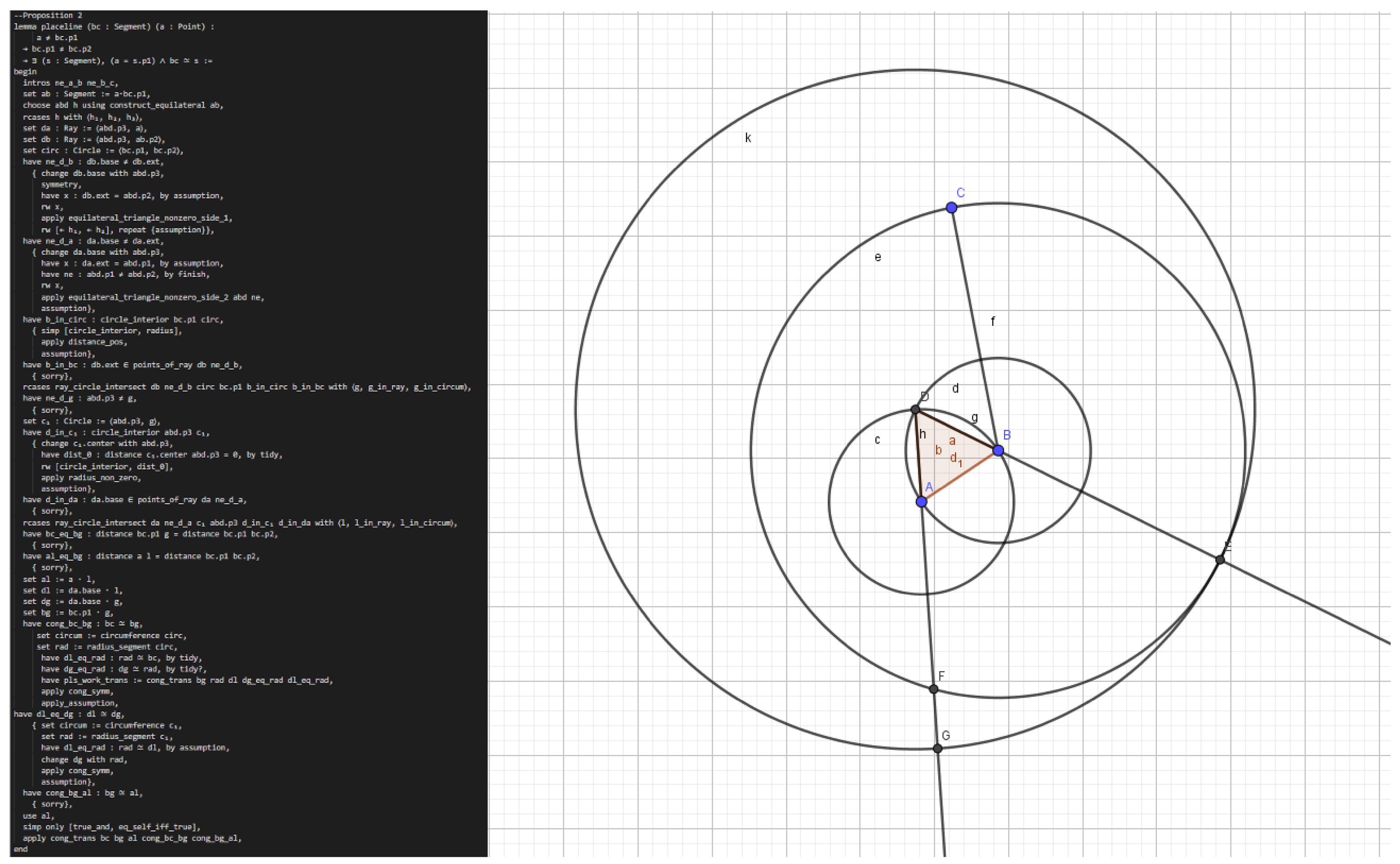

Figure 4: *(Constructing a segment congruent to another with a given endpoint). Example of a more verbose proof in Lean.*

Challenges

Limitations in Lean (mostly our own unfamiliarity with the cache-building system) as well as a lack of good online collaborative-editing platforms supporting Lean severely slowed our progress, and we were only able to prove a handful of the propositions we planned to prove. We were able to formalize all of Euclid's axioms and concepts of planar geometry, which could allow for future work and extensions to this project.

Hilbert's Geometry

Parallel to formalizing Euclid's work in Lean, we formalized Hilbert's axioms[4], with slight modifications, in Lean. For example, in other theorem prover formalizations of Hilbert's axioms[1], lines are defined as a fundamental type, but we chose to define lines as a *structure* from two points and a proof of distinctness.

In order to prove more complicated facts in Lean, we needed to formalize various structures and relations (such as **triangles** and a definition of **supplementary angles**) using Hilbert's primitive ideas.

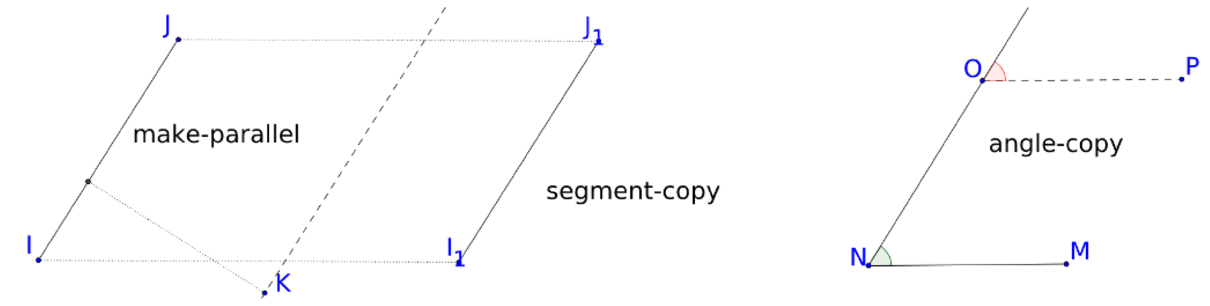

Figure 5: *(Euclid vs. Hilbert). Hilbert's geometry also differs from Euclid in these three extra postulated notions – one can construct a parallel line and copy a segment or angle.*

Hilbert's axioms create a **synthetic geometry** system, so he tends to avoid certain definitions (like **distance**).[3] We had to introduce these notions in ways that were compatible with his system.

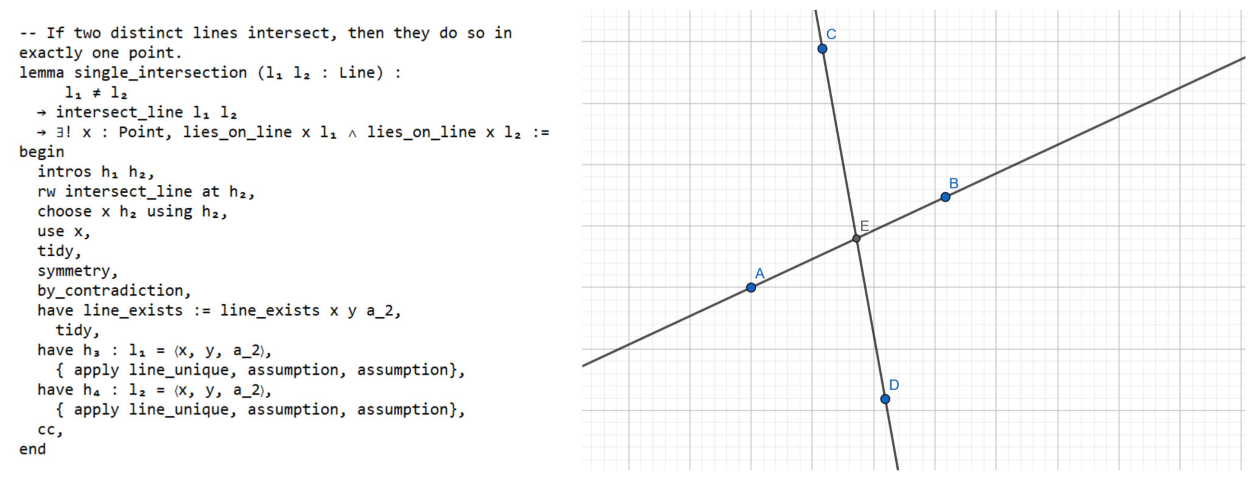

Figure 6: *(If two lines intersect, it must be at a single point). A proof using Lean's tactics and Hilbert's formalized axioms.*

Progress Report

We were able to formalize *all* of Euclid's and Hilbert's axioms and *most* of Tarski's axioms in Lean. We also proved 2 out of Euclid's 48 propositions (from Book 1, *Elements*) in the Euclidean system and several other lemmas in the Hilbertian system.

Future Directions

As a result of our short time-frame and limitations within Lean, we were unable to finish the proofs of many propositions in Book 1 of *Euclid's Elements*. However, we hope that our formalization so far can be utilized to prove many more complex theorems, such as the **Pythagorean Theorem** and a number of Euclid's other propositions. Euclid's elements aside, our work can act as the foundation to formalize other geometric structures, like **cyclic quadrilaterals** and **medians**.

References

[1] Gabriel Braun, Julien Narboux. *From Tarski to Hilbert.* Automated Deduction in Geometry 2012, Jacques Fleuriot, Sep 2012, Edinburgh, United Kingdom. pp.89-109, ff10.1007/978-3-642-40672-0_7ff.
[2] David E. Joyce. *Euclid's Elements, Book 1.* Clark University, Worchester, MA 01610
[3] K. Borsuk and Wanda Szmielew. (1960). *Foundations of geometry, Euclidean and Bolyai-Lobachevskian geometry: projective geometry.*
[4] E. J. Townsend, translator. *The Foundations of Geometry.* By David Hilbert, The Open Court Publishing Company, 1950.

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