

# NOTES ON HOMOLOGY

VAIBHAV KARVE

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## 1. DEFINITIONS

- (1) Let  $K$  : simplicial complex,  $p$  : integer dimension. A  $p$ -chain is a formal sum of  $p$ -simplices in  $K$ . Standard notation is  $c = \sum a_i \sigma_i$  where  $a_i$  are the coefficients (either 0 or 1 that is, modulo 2 coefficients).
- (2) Chains are added componentwise  $\sum a_i \sigma_i + b_i \sigma_i = \sum (a_i + b_i) \sigma_i$  where  $1 + 1 = 0$ . Denote by  $(C_p, +)$  the abelian group of  $p$ -chains with addition.
- (3) For  $p < 0$  and for  $p > \dim K$ , the group  $C_p(K)$  is trivial.
- (4) Define *boundary* :  $p$ -chain  $\rightarrow$   $(p - 1)$ -chain as the sum of the  $(p - 1)$ -dimensional faces, denoted by  $\partial_p c$ . Extend linearly so we have  $\partial_p : C_p \rightarrow C_{p-1}$ . This makes the boundary map a group homomorphism.
- (5) The *Chain complex* is the sequence of chain groups connected by boundary maps —

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots$$

- (6) A  $p$ -cycle is a  $p$ -chain with empty boundary that is,  $\partial_p c = 0$ . The  $p$ -cycles themselves form an abelian group denoted  $Z_p = Z_p(K) = \ker \partial_p$ . We have  $Z_p \leq C_p$ ,  $\forall p$  (subgroup).
- (7)  $Z_0 = \ker \partial_0 = C_0$ .
- (8) A  $p$ -boundary is a  $p$ -chain that is the boundary of a  $(p + 1)$ -chain,  $c = \partial d$  for  $d \in C_{p+1}$ . The  $p$ -boundaries also form an abelian group denoted  $B_p = \text{im } \partial_{p+1}$  which is also a subgroup of  $C_p$ .
- (9) *Fundamental lemma of homology*:  $\partial_p \partial_{p+1} d = 0$ , for every  $p$  : integer and every  $d \in C_{p+1}$ .  
Consequence: every  $p$ -boundary is also a  $p$ -cycle that is,  $B_p \leq Z_p$  (subgroup).

- (10) The  $p^{\text{th}}$  *homology group* is defined as the abelian group  $H_p = Z_p/B_p$ . The  $p^{\text{th}}$  *Betti number* is the rank of this group  $\beta_p = \text{rank } H_p$ . (the rank of a group is the smallest cardinality of a generating set of the group). So elements of  $H_p$  are of the form  $c + B_p$  where  $c \in Z_p$ . Each element is an equivalence class also called a *homology class*. Any two cycles in the same homology class are said to be *homologous*, denoted  $c \sim c'$ .

- (11)  $|C_p(K)| = 2^{\# \text{ of } p\text{-dimensional simplices in } K}$  and  $C_p \cong (\mathbb{Z}_2^n, \text{XOR})$ . Also have,

$$\text{ord } H_p = \text{ord } Z_p / \text{ord } B_p$$

and

$$\beta_p = \text{rank } H_p = \text{rank } Z_p - \text{rank } B_p$$

- (12) *Euler-Poincaré formula*: The Euler characteristic of a simplicial complex is the alternating sum of the number of simplices in each dimension. Let the rank of  $C_p, Z_p$  and  $B_p$  be denoted by  $n_p, z_p$  and  $b_p$  respectively. We have

$$\begin{aligned} \chi &= \sum_{p \geq 0} (-1)^p n_p \\ &= \sum_{p \geq 0} (-1)^p (z_p + b_{p-1}) \\ &\quad \text{(by Rank-nullity by viewing groups as a } \mathbb{Z}_2\text{-vspace)} \\ &= \sum_{p \geq 0} (-1)^p (z_p - b_p) \\ &= \sum_{p \geq 0} (-1)^p \beta_p \end{aligned}$$

- (13) The  $p^{\text{th}}$ -*boundary matrix* represents the  $(p-1)$ -simplices as rows and the  $p$ -simplices as columns (assuming arbitrary by fixed ordering of simplices). The  $(i, j)$ -entry is  $a_i^j = 1$  if the  $i^{\text{th}}$   $(p-1)$ -simplex is a face of the  $j^{\text{th}}$   $p$ -simplex and  $a_i^j = 0$  otherwise,  $\forall i \in \{1, \dots, n_{p-1}\}, \forall j \in \{1, \dots, n_p\}$ . Given a  $p$ -chain  $c = \sum a_k \sigma_k$ , the boundary  $\partial_p c$  can be computed by matrix multiplication of  $\partial_p = [a_i^j]$  with the column vector of coefficients  $(a_k)$

- (14) A collection of columns of  $\partial_p$  represents a  $p$ -chain and the sum of these columns gives its boundary. A collection of rows represents a  $(p-1)$ -chain and the sum of these rows gives its coboundary.

- (15) The rows of  $\partial_p$  form a basis of the group  $C_{p-1}$  and the columns form a basis of the group  $C_p$ .

- (16) *Smith normal form*: The matrix  $\partial_p$  (with entries in  $\mathbb{Z}_2$ ) is in Smith normal form if the initial segment of the diagonal is 1 (doesn't need to be the entire diagonal, just a segment) and everything else is 0.

$$\begin{bmatrix} \mathbf{I}_{b_{p-1} \times (n_p - z_p)} & \mathbf{O}_{b_{p-1} \times z_p} \\ \mathbf{O}_{(n_{p-1} - b_{p-1}) \times (n_p - z_p)} & \mathbf{O}_{(n_{p-1} - b_{p-1}) \times z_p} \end{bmatrix}_{n_{p-1} \times n_p}$$

Here, the leftmost  $b_{p-1}$  columns represent  $p$ -chains whose nonzero boundaries generate the group  $B_{p-1}$ . The rightmost  $z_p$  columns represent  $p$ -cycles that generate  $Z_p$  (these are zero columns in  $\partial_p$  but will become non-zero rows in  $\partial_{p+1}$ ).

- (17) Once all the boundary matrices (for different  $p$ ) are in Smith normal form, the Betti numbers can be calculated as differences between ranks —

$$\begin{aligned}\tilde{\beta}_p &= \text{rank} Z_p - \text{rank} B_p \\ &= z_p - b_p \\ &= (\# \text{ of zero columns in } N_p) - (\# \text{ of nonzero rows in } N_{p+1})\end{aligned}$$

where  $N_p$  is  $\partial_p$  in Smith normal form.

## 2. EXAMPLES

- (1)  $H_1(\text{Torus}) = \{B_1, x + B_1, y + B_1, x + y + B_1\}$  where  $B_1$  is the 1-boundary group, and  $x$  and  $y$  are the one non-bounding 1-cycles that go once around the arm and the hole of the torus.

Therefore,  $H_1(\text{Torus}) = \langle x + B_1, y + B_1 \rangle$  and hence  $\beta_1 = 2$ . This situation can be represented as a square with vertices  $\{0, x, y, x + y\}$ .

- (2) Closed ball is any triangulated topological space that is homeomorphic to  $\mathbb{B}^k = \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ .  $H_p(\text{Ball}) = 0, \forall p \geq 1$ . But  $\text{rank } H_0(\text{Ball}) = 1$  because  $H_0 \cong \mathbb{Z}_2$ .

If  $K$  is the set of faces of a single simplex of dimension  $k$  then  $\text{rank } C_p = \binom{k+1}{p+1}$ . For every  $c \in Z_p$ , one can find a  $d \in B_p$  such that  $\partial_p d = c$ . Hence  $Z_p = B_p$  and  $H_p = 0$  for  $p \geq 1$ .

- (3) If we build a tetrahedron one dimension at a time.

- (a) First there are only four vertices. Then  $\partial_0 = [1 \ 1 \ 1 \ 1]$ . Its normal form  $N_0$  has 1 nonzero row and 3 zero columns. Therefore  $\tilde{\beta}_0 = 3$ .

- (b) Now add the six vertices. Then  $\partial_0 = [1 \ 1 \ 1 \ 1]$  and  $\partial_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ .

$\partial_1$ 's normal form  $N_1$  has 3 nonzero rows and 3 nonzero columns.  $\tilde{\beta}_0 = 3(\text{from } N_0) - 3(\text{from } N_1) = 3 - 3 = 0$  and  $\tilde{\beta}_1 = 3(\text{from } N_1)$ .

- (c) At next step, we'll have  $\tilde{\beta}_0 = 3 - 3 = 0$ ,  $\tilde{\beta}_1 = 3 - 3 = 0$  and  $\tilde{\beta}_2 = 1$ .  
 (d) At the final step we'll have  $\tilde{\beta}_0 = 3 - 3 = 0$ ,  $\tilde{\beta}_1 = 3 - 3 = 0$ ,  $\tilde{\beta}_2 = 1 - 1 = 0$  and  $\tilde{\beta}_3 = 0$ .