NOTES ON ALGEBRAIC TOPOLOGY

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1. Some Underlying Geometric Notions

(1) Maps between spaces are always assumed to be continuous unless stated otherwise.

(2) A *deformation retraction of a space X onto a subspace A* is a family of maps $f_t : X \to X$ for $t \in I$ such that:
  (a) $f_0 = \mathbb{1}$ (the identity map),
  (b) $f_1(X) = A$, and
  (c) $f_t|A = \mathbb{1}$, for all $t \in I$.

(3) The family f_t should be continuous in the sense that the associated map $F : X \times I \to X$, given by $F(x, t) \mapsto f_t(x)$ is continuous.

(4) For a map $f : X \to Y$, the *mapping cylinder* M_f is the quotient space of the disjoint union $(X \times I) \sqcup Y$ obtained by identifying each $(x, 1) \in X \times I$ with $f(x) \in Y$.

(5) The mapping cylinder M_f deformation retracts to the subspace Y by sliding each point. Not all deformation retractions arise from mapping cylinders though.

(6) A *homotopy* is a family of maps $f_t : X \to Y$ for $t \in I$ such that the associated map $F : X \times I \to Y$ given by $F(x, t) = f_t(x)$ is continuous.

(7) Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are *homotopic* if $\exists$ a homotopy f_t connecting them. Then, one writes $f_0 \simeq f_1$.

(8) A *retraction of X onto A* is a map $r : X \to X$ such that $r(X) = A$ and $r|A = \mathbb{1}$. It can also be viewed instead as a map $r : X \to A$ such that $r|A = \mathbb{1}$. It can also be viewed as a map $r : X \to X$ such that $r^2 = r$. This makes retraction maps the topological analogs of projection operators.

(9) Hence, a deformation retraction of X onto subspace A is a homotopy from the identity map of X to a retraction of X onto A.

(10) Not all retractions come from deformation retractions. Any space X can be retracted to any single point $x_0 \in X$ via the constant map sending all of X to x_0.

(11) A space X that deformation retracts onto a point must necessarily be path-connected.

(12) A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is a *homotopy relative to* A. Thus, a deformation retraction of X onto A is a homotopy relative to A from the identity map of X to a retraction of X onto A.

(13) If a space X deformation retracts onto a subspace A via $f_t : X \to X$, then if $r : X \to A$ denotes the resulting retraction and $i : A \to X$ denotes the inclusion map, then we have $ri = \mathbb{1}$ (on A) and $ir \simeq \mathbb{1}$ (on X), the latter homotopy being given by f_t.

(14) A map $f : X \to Y$ is *homotopy equivalence* if there is a map $g : Y \to X$ such that $fg \simeq \mathbb{1}$ (on Y) and $gf \simeq \mathbb{1}$ (on X). In that case, the spaces X and Y are *homotopically equivalent* and have the same *homotopy type*, denoted $X \simeq Y$.

(15) If subspaces A, B and C are all deformation retractions of the same space X then they are homotopically equivalent. However, they need not be deformation retractions of each other.

(16) Two spaces X and Y are homotopically equivalent iff $\exists$ a third space Z containing both X and Y as deformation retracts. This is proved by choosing $Z = M_f$ for the homotopy equivalence $f : X \to Y$.

(17) A map is *nullhomotopic* if it is homotopic to a constant map.

(18) A space having the homotopy type of a point is *contractible*. In this case, the identity map of the space is nullhomotopic.

(19) An orientable surface M_g of genus g can be constructed from a polygon with $4g$ sides by identifying pairs of edges.

(20) Construction of a space X using cell-complexes can be done as follows:
   (a) Start with a discrete set X^0, whose points are regarded as 0-*cells*.
   (b) Inductively, form the n-*skeleton* X^n from X^{n-1} by attaching n-*cells* e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ that is, of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$.
   (c) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case, X is given the weak topology: a set $A \subseteq X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

A space X constructed this way is a *cell complex* or *CW complex*.

(21) The *dimension* of a cell complex X is the maximum of the dimensions of the cells of X.

(22) A one-dimensional cell complex is a *graph*.

(23) The *Euler characteristic* for a cell complex with finite number of cells is the number of even-dimensional cells minus the number of odd-dimensional cells. Euler characteristic is an invariant of homotopy types.

(24) The sphere S^n has the structure of a cell complex with two cells : e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.

(25) Real projective n-space $\mathbb{R}\mathrm{P}^n$ is obtained from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching an n-cell with the quotient projection $S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ as the attaching map. Hence, $\mathbb{R}\mathrm{P}^n$ has a cell complex structure $e^0 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.

(26) $\mathbb{R}\mathrm{P}^\infty = \bigcup_n \mathbb{R}\mathrm{P}^n$ becomes a cell complex with one cell in each dimension. Can also be viewed as the space of lines through the origin in $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$.

(27) Complex projective n-space $\mathbb{C}\mathrm{P}^n$ can be obtained form $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a cell e^{2n} via the quotient map $S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$. Hence, it has cell structure $e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions. Similarly, $\mathbb{C}\mathrm{P}^\infty$ has a cell structure with one cell in each even dimension.

(28) Each cell e^n_α in a cell complex X has a *characteristic map* $\phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α. This map tells us which points were identified in the construction of the cell complex of X.

(29) A *subcomplex* of a cell complex X is a closed subspace A of X that is a union of cells of X. Subcomplexes $\mathbb{R}\mathrm{P}^k \subseteq \mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^k \subseteq \mathbb{C}\mathrm{P}^n$ are the only subcomplexes for the real and complex projective spaces.

(30) A pair (X, A) consisting of a cell complex X and a subcomplex A of X is a *CW pair.*

(31) In general, the closure of each cell, or similarly, the closure of any collection of cells need not be a subcomplex.

(32) Operations on spaces:
   (a) **Product:** If X and Y are cell complexes, then $X \times Y$ has the structure of a cell complex with its cells being the products $e^m_\alpha \times e^n_\beta$, where e^m_α ranges over the cells of X and e^n_β ranges over the cells of Y.